MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINT
MATHEMATICS
DIR. FIROZ AHMAD
M.Sc. (Maths), B.Ed, M.Phil (Maths)

RAM RAJYA MORE, SIWAN

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPATETIVE EXAM FOR XII (PQRS)

DEFINITE INTEGRATION

& Their Properties

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THINGS TO REMEMBER

✵ Definite Integration

Let $f(x)$ be a continuous function defined on a closed interval [a, b] and $f(x)$ dx = F(x) + c, then

$$\int_a^b f(x)dx = [F(x)]_a^b$$

or $$\int_a^b f(x)dx = F(b) - F(a)$$

The numbers a and b are called the limits of integration. a is called the lower limit and b is called the upper limit.

Geometrical Interpretation of a definite integral

Geometrically it represents an algebraic sum of the areas of regions bounded by graph of the function y = $f(x)$, the x-axis ad the straight lines x = a and x = b. The areas above x-axis are taken as positive and the areas below x-axis are taken as negative.

$$\int_a^b f(x)dx = A_1 - A_2 + A_3 - A_4 + A_5$$

Where A_1, A_2, A_3, A_4 and A_5 are the areas of the shaded region.

✵ Properties of Definite Integration

1. $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_z^b f(u)du$

   Here, x is a dummy variable, it can be replaced by any other variable t, u, ...

   ie, The value of a definite integral does not change with change of variable of integration provided the limits of integration remains the same.

2. Interchanging the limits of the definite integral does not change the absolute value but change the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3. $\int_a^b f(x)dx = \int_a^b f(x)dx + \int_c^b f(x)dx, (a < c < b)$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

   In particular, $\int_a^b f(x)dx = \int_a^b f(a-x)dx$

   Special Case

$$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$$

$$= \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}dx$$

$\Rightarrow \qquad 2\text{I} = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}dx$

$\Rightarrow \qquad 2\text{I} = \int_a^b dx = (b-a)$

$\therefore \qquad \text{I} = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2} \qquad$ ...(i)

Eq. (i) is a special case 4th property and is used as standard result.

5. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$ (In general)

$$= \begin{cases} 2\int_0^a f(x)dx, & \text{If } f(2\text{a}-x) = f(x) \\ 0, & \text{If } f(2\text{a}-x) = f(x) \end{cases}$$

6. $$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{If } f(-x) = f(x) \\ & \text{ie, } f(x) \text{ is even.} \\ 0, & \text{If } f(-x) = -f(x) \\ & \text{ie, } f(x) \text{ is odd.} \end{cases}$$

7. **Leibnitz's Rule**

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = \text{h}'(x)f[\text{h}(x)] - \text{g}'(x)f[\text{g}(x)]$$

In particular $\frac{d}{dx}\int_a^{h(x)} f(t)dt\, \text{h}'(x)f[\text{h}(x)]$,

{a is any constant independent of x}

or $\qquad \frac{d}{dx}\int_a^x f(t)dt = f(x)$

8. If $f(x)$ is a periodic function with period T, then

(a) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in I$

(b) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx,\ n \in I,\ a \in R$

(c) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx,\ m,\ n \in I$

(d) $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx,\ n \in I,\ a \in R$

(e) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx,\ n \in I, a, b \in R$

9. If $f(x) \geq \text{g}(x)$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ (where b > a.)

10. If $f(x) \geq 0$ for all $x \in$ [a, b], then $\int_a^b f(x)dx \geq 0$.

11. $|\int_a^a f(x)dx| \leq \int_a^a |f(x)|\, dx$

12. If m and M are global minima and global maxima of $f(x)$ in [a, b], ie, m < $f(x)$ M for a < x < b, then

✷ Definite Integration by Substitution

When we substitute, we are chaging the variable, so we cannot use the same upper and lower limits. We can either

do the problem asan indefinite first, the use upper and lower limits later.

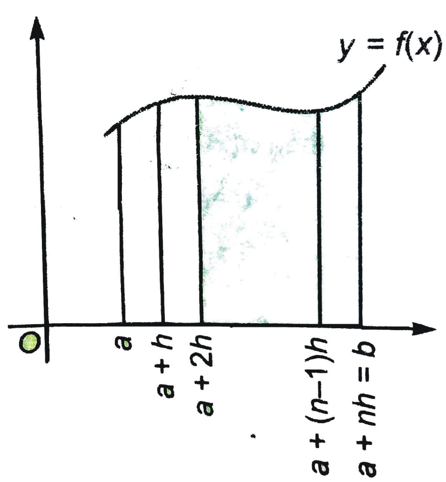
Or do the problem throughout using the new variable and the new upper and lower limits.

Show the correct variable for the upper and lower limit during the substitution phase.

✷ Definite Integral as a Limit of Sum

Let $f(x)$ be a continuous real valued function defined on the closed interval [a, b] which is devided into n parts. The point of division on x-axis are a, a + h, a + 2h,......a + (n – 1)h, a + nh where h = $\frac{b-a}{n}$.

Let S_n denotes the area of these n rectangles. Then S_n = hf(a) + hf(a + h) + hf(a + 2h)+.....+hf(a + (n – 1)h)



Clearly S_n is area very close to the area of the region bounded by curve y =$f(x)$, x-axis and the ordinates x = a, x = b.

Hence, $$\int_a^b f(x)dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{r=0}^{n-1} hf(a+rh)$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)r}{n}\right)$$

In particular, if a = 0, b = 1, then

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right).$$

✱ Walli's Formula

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \begin{cases} (m-1)(m-3)......\dfrac{(n-1)(n-3).......}{(m+n)(m+n-2)......}\times\dfrac{\pi}{2}, & \text{when both m, n are even} \\ (m-1)(m-3)......\dfrac{(n-1)(n-3).......}{(m+n)(m+n-2)(m+n-4)......}, & \text{Otherwise} \end{cases}$$

Note :

- In a definite integral, there is no need to keep the constant of integration.
- The indefinite integral $\int f(x)$ dx is a function of x, whereas definite intergral $\int_a^b f(x)$ dx is a number.
- Given $\int f(x)$ dx we can find $\int_a^b f(x)$ dx, but given $\int_a^b f(x)$ dx we cannot find $\int f(x)$ dx.
- If f(t) is an odd function, then $\phi(x) = \int_0^x f(t)$dt is an even function.
- If f(t) is an even function, then $\phi(x) = \int_0^x f(t)$dt is an odd function.
- If f(t) is is discontinuous at x = a, then

$$\int_0^{2a} f(x)dx = \int_0^a [f(a-x)+f(a+x)]\, dx$$

- we can also write

S_n= hf(a + h) + hf(a + 2h)+............+hf(a + nh)

and $$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n} hf(a+rh),\ h=\frac{b-a}{n}$$

- To express the limit of sum as definite integral replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\lim\limits_{n\to\infty}\sum$ **by** $\int$

For limit, evaluate $\lim\limits_{n\to\infty}\left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

eg, $$\lim_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_o^p f(x)dx$$

$\therefore$ $$\left[\lim_{n\to\infty}\left(\frac{r}{n}\right)\right]_{r=1} = 0,\quad \left[\lim_{n\to\infty}\left(\frac{r}{n}\right)\right]_{r=np} = p$$